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On the system of partial differential equations associated with Appell's function F_4

Harold Exton
'Nyuggel', Lunabister, Dunrossness, Shetland ZE2 9JH, UK

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Abstract. The fourth Appell function of two variables arises in certain aspects of mathematical physics. All the solutions of the associated system of partial differential equations have not so far been given explicitly. Analytic continuation formulae of the function F_4 have only been partially recorded. Some of these gaps have been filled in this study.

1. Introduction

Over one hundred years ago, Appell (1880) introduced four double hypergeometric functions as generalizations of the ordinary hypergeometric function ${}_2F_1$. Perhaps the most intractable of the Appell functions is the fourth, defined as

$$F_4(a, b; c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a, m+n)(b, m+n)}{(c, m)(c', n)m!n!} x^m y^n \qquad |x|^{1/2}+|y|^{1/2}<1. \tag{1.1}$$

As usual,

$$(a, m) = a(a+1)(a+2)\dots(a+m-1) = \Gamma(a+m)/\Gamma(a) \qquad (a, 0) = 1. \tag{1.2}$$

All the Appell functions occur in a number of contexts related to mathematical physics. For applications of F_4, see Grant and Quiney (1993), Kalnins, Manocha and Miller (1980), Sack (1964) and Appell and Kampé de Fériet (1926) (page 292) for example.

These papers include applications in the areas of particle physics, Lie theory and the four-dimensional wave equation, two-centre expansions of the distance between two points and celestial mechanics, respectively.

As pointed out by Kalnins, Manocha and Miller (1980), the 34 distinct second-order hypergeometric functions of two variables listed by Horn (1931) arise in connection with the four-variable wave equation in several coordinate systems. In some cases, the three-variable wave and heat equations and the two-variable Helmholz equation also occur. Considerably more interest attaches to these functions than had previously been recognized.

The properties of many of the 34 double hypergeometric functions throughout the whole plane can be dealt with by considering the system of partial differential equations associated with the first Appell function which has been discussed in great detail by Erdélyi (1950) from the point of view of analysing its contour integral solutions.

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Erdélyi (1953) (page 237) also points out that every system associated with a double hypergeometric series of the second order with the exception of the systems of the functions of Horn's list

$$F_4, \quad H_1, \quad H_5$$

and a confluent form of H_1, can be reduced to a system associated with the second Appell function or to a particular or limiting case of it. The system F_2 has been treated exhaustively by Olsson (1977) and the behaviour of its solutions over the whole plane has been given in detail. A clear need arises for an investigation of the function F_4 from this point of view. The functions H_1 and H_5 should also be investigated.

In order to consider the behaviour of a function such as F_4 outside the boundary of convergence of its series representation (1.1), an investigation of the associated system of partial differential equations is carried out. In this case, the system is

$$x(1-x)\frac{\partial^2 F}{\partial x^2} - y^2\frac{\partial^2 F}{\partial y^2} - 2xy\frac{\partial^2 F}{\partial x\,\partial y} + [c-(a+b+1)x]\frac{\partial F}{\partial x} - (a+b+1)y\frac{\partial F}{\partial y} - abF = 0$$

$$y(1-y)\frac{\partial^2 F}{\partial y^2} - x^2\frac{\partial^2 F}{\partial x^2} - 2xy\frac{\partial^2 F}{\partial x\,\partial y} + [c'-(a+b+1)y]\frac{\partial F}{\partial y} - (a+b+1)x\frac{\partial F}{\partial x} - abF = 0 \qquad (1.3)$$

which has been studied in a general way by Erdélyi (1941) where all the singularities of (1.3) were discussed and the associated types of solutions outlined by means of a contour integral representation of F_4. Erdélyi, however, says little in the way of explicit solutions of (1.3) or of the connections between them. It is shown that the singularities of the system consist of two sets. The first set is the points $(0, 0)$, $(0, \infty)$ and $(\infty, 0)$ each formed by the intersection of two singular manifolds of (1.3). The second set of singular points, of a more complicated nature than the first, consists of the points $(0, 1)$, $(1, 0)$ and (∞, ∞). In this case, each singularity is formed from the contact of two singular manifolds. For a more detailed description of this aspect of the relevant analysis, the reader is referred to Erdélyi (1941). Furthermore, Appell (1880) has shown that the general integral of the system (1.3) may be represented as the linear combination of four linearly independent integrals near each singular point.

The method of tackling the problem adopted in this paper consists of expanding known solutions of (1.3) in series of hypergeometric functions ${}_2F_1$ and applying analytical continuations of these latter functions. While this method has been anticipated by Appell and Kampé de Fériet (1926), it remained for Olsson (1964) to deduce explicit solutions and relations between them of the partial differential system of the Appell function F_1 by this means.

While transformations of the associated system of partial differential equations and its integral solutions would yield the required results, the approach adopted here gives the analytic continuation formulae in a more straightforward way.

In his study of the Appell functions F_2 and F_3, Olsson (1977) also uses the method of corresponding solutions of a relevant system of partial differential equations, whereby any knowledge of the solution set can be extended in a straightforward manner.

Up to the present time, considerable difficulty has been experienced in dealing with the function F_4. In what follows, any values of the parameters for which any expressions do not make sense are tacitly excluded.

2. Corresponding solutions of (1.3)

Appell and Kampé de Fériet (1926) (page 52) have given the four independent solutions of (1.3) valid in the whole neighbourhood of the origin in the form

$$F_4(a, b; c, c'; x, y) \tag{2.1}$$

$$x^{1-c}F_4(a+1-c, b+1-c; 2-c, c'; x, y) \tag{2.2}$$

$$y^{1-c'}F_4(a+1-c', b+1-c'; c, 2-c'; x, y) \tag{2.3}$$

and

$$x^{1-c}y^{1-c'}F_4(a+2-c-c', b+2-c-c'; 2-c, 2-c'; x, y). \tag{2.4}$$

Furthermore, we have the solutions

$$y^{-a}F_4(a, a+1-c'; c, a+1-b; x/y, 1/y) \tag{2.5}$$

and

$$y^{-b}F_4(b+1-c', b; c, b+1-a; x/y, 1/y). \tag{2.6}$$

See Appell and Kampé de Fériet (1926) (page 26).
Hence, if

$$Z(a, b; c, c'; x, y) \tag{I}$$

is a solution of (1.3), then, by symmetry, so also are

$$Z(b, a; c, c'; x, y) \tag{II}$$

and

$$Z(a, b; c', c; y, x). \tag{III}$$

From (2.2) to (2.4), we have the further general solution types

$$x^{1-c}Z(a+1-c, b+1-c; 2-c, c'; x, y) \tag{IV}$$

$$y^{1-c'}Z(a+1-c', b+1-c'; c, 2-c'; x, y) \tag{V}$$

and

$$x^{1-c}y^{1-c'}Z(a+2-c-c', b+2-c-c'; 2-c, 2-c'; x, y). \tag{VI}$$

The solutions (2.5) and (2.6) also imply that

$$y^{-a}Z(a, a+1-c'; c, a+1-b; x/y, 1/y) \tag{VII}$$

and

$$y^{-b}Z(b+1-c', b; c, b+1-a; x/y, 1/y) \tag{VIII}$$

are solutions of (1.3).

The above solution forms may be used in any combination. However, whether or not any results so obtained are linearly independent must be investigated separately in each individual case. This technique is very useful and straightforward to apply. See also Exton (1992).

3. Expansion in terms of Gauss functions

The function F_4 as defined by (1.1) may be expanded in series of Gauss functions as follows:

$$F_4(a, b; c, c'; x, y) = \sum_{m=0}^{\infty} \frac{(a, m)(b, m)}{(c, m)m!} x^m {}_2F_1(a+m, b+m; c'; y). \tag{3.1}$$

If the inner ${}_2F_1$ function on the right of (3.1) is replaced by any of its analytic continuations, then each term of the result is also a solution of the system of partial differential equations (1.3). This follows from the fact that this process is equivalent to taking the Barnes integral

$$F_4(a, b; c, c'; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)(-x)^t}{\Gamma(c+t)} {}_2F_1(a+t, b+t; c'; y)\, dt \tag{3.2}$$

(Appell and Kampé de Fériet (1926) page 40) and replacing the Gauss function of the integrand as above.

The solutions (2.5) and (2.6), valid near the point $(0, \infty)$, were, in fact, obtained by Appell and Kampé de Fériet (1926) by applying the formula

$${}_2F_1(a, b; c; y) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-y)^{-a} {}_2F_1(a, a-c+1; a-b+1; 1/y) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-y)^{-b} {}_2F_1(b, b-c+1; b-a+1; 1/y). \tag{3.3}$$

See, for example, Erdélyi (1953) (page 108). The result

$$F_4(a, b; c, c'; x, y) = \frac{\Gamma(c')\Gamma(b-a)}{\Gamma(c'-a)\Gamma(b)} (-y)^{-a} F_4(a, a-c'+1; c, a+1-b; x/y, 1/y) + \frac{\Gamma(c')\Gamma(a-b)}{\Gamma(c'-b)\Gamma(a)} (-y)^{-b} F_4(b, b-c'+1; c, b+1-a; x/y, 1/y) \tag{3.4}$$

is then obtained. This expression will be used in the following.

Fundamental systems of solutions valid in the whole neighbourhood of the points $(0, \infty)$ and $(\infty, 0)$ may be deduced from the appropriate corresponding solutions (I) to (VIII). These solutions are already well known.

4. Solutions near the point (0, 1)

In the first instance, solutions near this point are tackled by means of the continuation formula

$$ {}_2F_1(a,b;c;y)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b;a+b-c+1;1-y) $$

$$ +\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-y)^{c-a-b}\,{}_2F_1(c-a,c-b;c-a-b+1;1-y) \quad (4.1) $$

(Erdélyi (1953) page 108).

If we apply (4.1) to the expansion (3.1), after a little algebra, the first term of the result gives the solution

$$ \frac{\Gamma(c')\Gamma(c'-a-b)}{\Gamma(c'-a)\Gamma(c'-b)}\sum_{m,n=0}^{\infty}\frac{(a,m+n)(b,m+n)(1+a-c',m)(1+b-c',m)x^m(1-y)^n}{(a+b-c'+1,2m+n)(c,m)m!n!} \quad (4.2) $$

written for convenience as

$$ \frac{\Gamma(c')\Gamma(c'-a-b)}{\Gamma(c'-a)\Gamma(c'-b)}G(a,b,1+a-c',1+b-c';a+b-c'+1,c;x,1-y). \quad (4.3) $$

This series converges in the whole neighbourhood of the singular point (0, 1).

The second term of (4.1) gives rise to a type of solution of (1.3) which is quite different in character. After a little reduction, we obtain the solution

$$ \frac{\Gamma(c')\Gamma(a+b-c')}{\Gamma(a)\Gamma(b)}(1-y)^{c'-a-b}\sum_{m,n=0}^{\infty}\frac{(a+b-c',2m-n)(1+a-c',m)(1+b-c',m)}{(1+a-c',m-n)(1+b-c',m-n)(c,m)m!n!} $$

$$ \times\left[\frac{x}{(1-y)^2}\right]^m(y-1)^n \quad (4.4) $$

or, for convenience,

$$ \frac{\Gamma(c')\Gamma(a+b-c')}{\Gamma(a)\Gamma(b)}(1-y)^{c'-a-b} $$

$$ \times H\left(a+b-c',1+a-c',1+b-c';1+a-c',1+b-c',c;\frac{x}{(1-y)^2},y-1\right). \quad (4.5) $$

Unlike (4.2), this last series converges in only that portion of the neighbourhood of the point (0, 1) for which $|1-y|^2>|x|$. This is to be expected from the general theory outlined by Erdélyi (1941) in which the more complicated singularities (0, 1), (1, 0) and (∞,∞) were characterized by the fact that a complete fundamental system of solutions of (1.3) cannot be found such that all of its members converge in the whole neighbourhood of the singularity in question.

In order to seek solutions convergent near (0, 1) when $|x|>|1-y|^2$, we expand (4.4) as a series of fourth-order hypergeometric functions, namely

$$\sum_{n=0}^{\infty} \frac{(c'-a,n)(c'-b,n)}{(c'-a-b+1,n)n!}(1-y)^n \times {}_4F_3\left[\begin{matrix} \frac{a}{2}+\frac{b}{2}-\frac{c'}{2}-\frac{n}{2}, \frac{a}{2}+\frac{b}{2}-\frac{c'}{2}+\frac{1}{2}-\frac{n}{2}, 1+a-c', 1+b-c'; \frac{4x}{(1-y)^2} \\ c, 1+a-c'-n, 1+b-c'-n \end{matrix}\right]. \tag{4.6}$$

The analytic continuation of the inner ${}_4F_3$ series for large values of its argument in furnished by considering its representation as a Meijer G-function in the form

$${}_4F_3\left(\begin{matrix} a_1, a_2, a_3, a_4; \\ b_1, b_2, b_3 \; ; \end{matrix} x\right) = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)} G^{4,1}_{4,4}\left(-\frac{1}{x}\middle| \begin{matrix} 1, b_1, b_2, b_3 \\ a_1, a_2, a_3, a_4 \end{matrix}\right) \tag{4.7}$$

and expanding the G-function in the usual way. See Erdélyi (1953) (pages 208 and 215). The required continuation takes the form

$$\begin{aligned}
&\frac{\Gamma(c)\Gamma(1+a-c'-n)\Gamma(1+b-c'-n)\Gamma(\frac{1}{2})\Gamma(1+\frac{a}{2}-\frac{b}{2}-\frac{c'}{2}+\frac{n}{2})\Gamma(1-\frac{a}{2}+\frac{b}{2}-\frac{c'}{2}+\frac{n}{2})}{\Gamma(\frac{a}{2}+\frac{b}{2}-\frac{c'}{2}+\frac{1}{2}-\frac{n}{2})\Gamma(1+a-c')\Gamma(1+b-c')\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{c'}{2}+\frac{n}{2})\Gamma(1+\frac{a}{2}-\frac{b}{2}-\frac{c'}{2}-\frac{n}{2})\Gamma(1-\frac{a}{2}+\frac{b}{2}-\frac{c'}{2}-\frac{n}{2})} \\
&\quad\times\left[\frac{-4x}{(1-y)^2}\right]^{\frac{c'}{2}-\frac{a}{2}-\frac{b}{2}+\frac{n}{2}} {}_4F_3\left[\begin{matrix} \frac{a}{2}+\frac{b}{2}-\frac{c'}{2}-\frac{n}{2}, 1+\frac{a}{2}+\frac{b}{2}-c-\frac{c'}{2}-\frac{n}{2}, \frac{b}{2}-\frac{a}{2}+\frac{c'}{2}+\frac{n}{2}, \frac{a}{2}-\frac{b}{2}+\frac{c'}{2}+\frac{n}{2}; \frac{(1-y)^2}{4x} \\ \frac{1}{2}, \frac{b}{2}-\frac{a}{2}+\frac{c'}{2}-\frac{n}{2}, \frac{a}{2}-\frac{b}{2}+\frac{c'}{2}-\frac{n}{2} \; ; \end{matrix}\right] \\
&\quad+\frac{\Gamma(c)\Gamma(1+a-c'-n)\Gamma(1+b-c'-n)\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2}+\frac{a}{2}-\frac{b}{2}-\frac{c'}{2}+\frac{n}{2})\Gamma(\frac{1}{2}-\frac{a}{2}+\frac{b}{2}-\frac{c'}{2}+\frac{n}{2})}{\Gamma(\frac{a}{2}+\frac{b}{2}-\frac{c'}{2}-\frac{n}{2})\Gamma(1+a-c')\Gamma(1+b-c')\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{c'}{2}-\frac{1}{2}+\frac{n}{2})\Gamma(\frac{1}{2}+\frac{a}{2}-\frac{b}{2}-\frac{c'}{2}-\frac{n}{2})\Gamma(\frac{1}{2}-\frac{a}{2}+\frac{b}{2}-\frac{c'}{2}-\frac{n}{2})} \\
&\quad\times\left[\frac{-4x}{(1-y)^2}\right]^{\frac{c'}{2}-\frac{a}{2}-\frac{b}{2}-\frac{1}{2}+\frac{n}{2}} \\
&\quad\times {}_4F_3\left[\begin{matrix} \frac{a}{2}+\frac{b}{2}-\frac{c'}{2}+\frac{1}{2}-\frac{n}{2}, \frac{3}{2}+\frac{a}{2}+\frac{b}{2}-c-\frac{c'}{2}-\frac{n}{2}, \frac{1}{2}+\frac{b}{2}-\frac{a}{2}+\frac{c'}{2}+\frac{n}{2}, \frac{1}{2}+\frac{a}{2}-\frac{b}{2}+\frac{c'}{2}+\frac{n}{2}; \frac{(1-y)^2}{4x} \\ \frac{3}{2}, \frac{1}{2}-\frac{a}{2}+\frac{b}{2}+\frac{c'}{2}-\frac{n}{2}, \frac{1}{2}+\frac{a}{2}-\frac{b}{2}+\frac{c'}{2}-\frac{n}{2} \; ; \end{matrix}\right].
\end{aligned} \tag{4.8}$$

Two of the terms normally expected in the analytic continuation of the general function ${}_4F_3$ vanish in this particular case. For convenience, a further double hypergeometric function of higher is introduced as

$$K(a,b,c,d;f,g,h,k;x,y) \sum_{m,n=0}^{\infty} \frac{(a,m+n)(b,m+n)(c,m-n)(d,m-n)x^m y^n}{(f,m-n)(g,m-n)(h,m)(k,n)m!n!}. \tag{4.9}$$

After some rather tedious algebra, an additional solution of (1.3) is obtained, namely

$$\begin{aligned}
&\frac{\Gamma(c')\Gamma(a+b-c')}{\Gamma(a)\Gamma(b)}\Gamma(c)\Gamma(\tfrac{1}{2})(-4x)^{\frac{c'}{2}-\frac{a}{2}-\frac{b}{2}} \\
&\quad\times\left[\frac{1}{\Gamma(\frac{a}{2}+\frac{b}{2}-\frac{c'}{2}+\frac{1}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{c'}{2})} \times K\left(\begin{matrix} \frac{b}{2}-\frac{a}{2}+\frac{c'}{2}, \frac{a}{2}-\frac{b}{2}+\frac{c'}{2}, \frac{a}{2}+\frac{b}{2}-\frac{c'}{2}, \frac{a}{2}+\frac{b}{2}-c-\frac{c'}{2}+1; \frac{(1-y)^2}{4x}, \frac{x}{4} \\ \frac{b}{2}-\frac{a}{2}+\frac{c'}{2}, \frac{a}{2}-\frac{b}{2}+\frac{c'}{2}, \frac{1}{2}, \frac{1}{2} \; ; \end{matrix}\right)\right. \\
&\quad+\frac{(c'+a-b-1)(c'-a+b-1)(-x)^{1/2}}{(c'-a-b+1)\Gamma(\frac{a}{2}+\frac{b}{2}-\frac{c'}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{c'}{2}+\frac{1}{2})} \\
&\quad\left.\times K\left(\begin{matrix} \frac{b}{2}-\frac{a}{2}+\frac{c'}{2}+\frac{1}{2}, \frac{a}{2}-\frac{b}{2}+\frac{c'}{2}+\frac{1}{2}, \frac{a}{2}+\frac{b}{2}-\frac{c'}{2}, \frac{a}{2}+\frac{b}{2}-c-\frac{c'}{2}+\frac{1}{2}; \frac{(1-y)^2}{4x}, \frac{x}{4} \\ \frac{b}{2}-\frac{a}{2}+\frac{c'}{2}-\frac{1}{2}, \frac{a}{2}-\frac{b}{2}+\frac{c'}{2}-\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \; ; \end{matrix}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Gamma(c')\Gamma(a+b-c')}{\Gamma(a)\Gamma(b)}\Gamma(c)\Gamma(-\tfrac{1}{2})(1-y)(-4x)^{\frac{c'}{2}-\frac{a}{2}-\frac{b}{2}} \\
&\times\Bigg[\frac{1}{\Gamma(\frac{a}{2}+\frac{b}{2}-\frac{c'}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{c'}{2}-\frac{1}{2})} \\
&\quad\times K\left(\begin{matrix}\frac{b}{2}-\frac{a}{2}+\frac{c'}{2}+\frac{1}{2},\ \frac{a}{2}-\frac{b}{2}+\frac{c'}{2}+\frac{1}{2},\ \frac{a}{2}+\frac{b}{2}-\frac{c'}{2}+\frac{1}{2},\ \frac{a}{2}+\frac{b}{2}-c-\frac{c'}{2}+\frac{3}{2}; \\ \frac{b}{2}-\frac{a}{2}+\frac{c'}{2}+\frac{1}{2},\ \frac{a}{2}-\frac{b}{2}+\frac{c'}{2}+\frac{1}{2},\ \frac{3}{2},\ \frac{1}{2} \qquad ;\end{matrix}\ \frac{(1-y)^2}{4x},\frac{x}{4}\right) \\
&+\frac{(c'+a-b)(c'-a+b)(-x)^{1/2}}{(c'-a-b+1)\Gamma(\frac{a}{2}+\frac{b}{2}-\frac{c'}{2}-\frac{1}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{c'}{2})} \\
&\quad\times K\left(\begin{matrix}\frac{b}{2}-\frac{a}{2}+\frac{c'}{2}+1,\ \frac{a}{2}-\frac{b}{2}+\frac{c'}{2}+1,\ \frac{a}{2}+\frac{b}{2}-\frac{c'}{2},\ \frac{a}{2}+\frac{b}{2}-c-\frac{c'}{2}+1; \\ \frac{b}{2}-\frac{a}{2}+\frac{c'}{2},\ \frac{a}{2}-\frac{b}{2}+\frac{c'}{2},\ \frac{3}{2},\ \frac{3}{2} \qquad ;\end{matrix}\ \frac{(1-y)^2}{4x},\frac{x}{4}\right)\Bigg].
\end{aligned}
\tag{4.10}
$$

Let us denote the expression (4.10) by $L(a, b; c, c'; \frac{1}{4}(1-y)^2/x, \frac{1}{4}x)$. This series converges in that portion of the neighbourhood of the singular point (0, 1) for which $|4x|>|1-y|^2$.

5. The singular point (∞, ∞)

Taking the corresponding solutions of section 2, it follows that, by applying (III) to (VII), we have the further solution type

$$y^{-a}Z(a, a+1-c'; a+1-b, c; 1/y, x/y). \tag{IX}$$

We then apply this expression to (4.10) and obtain the solution

$$y^{-a}L(a, a+1-c'; a+1-b, c; \tfrac{1}{4}(x-y)^2/y, \tfrac{1}{4}/y). \tag{5.1}$$

It will be seen that the form of the solution (5.1) converges in the whole neighbourhood of the singularity (∞, ∞). A second independent solution of this type follows from (VIII), namely

$$y^{-b}L(b+1-c', b; b+1-a, c; \tfrac{1}{4}(x-y)^2/y, \tfrac{1}{4}/y). \tag{5.2}$$

Independent solutions convergent in that part of the neighbourhood of the point (∞, ∞) for which $|y|>|x|$ follow from (IV) as applied to (VII) and (VIII), respectively:

$$x^{1-c}y^{c-1-a}F_4(a+1-c, a+2-c-c'; 2-c, a+1-b; x/y, 1/y) \tag{5.3}$$

and

$$x^{1-c}y^{c-1-b}F_4(b+1-c, b+2-c-c'; 2-c, b+1-a; x/y, 1/y). \tag{5.4}$$

A pair of solutions similarly applicable when $|x|>|y|$ may be deduced by interchanging (x, c) with (y, c') in the above.

Hence, complete fundamental systems of solutions relative to all the singularities of the partial differential system (1.3) can be written down. Furthermore, all these solutions can be expressed as double hypergeometric series. In the case of the points (0, 1), (1, 0) and (∞, ∞), the series G, H and K are new.

6. The convergence of the new series representations

If the technique of cancellation of parameters is applied to the series G (equation (4.3)) then this converges with

$$\sum_{m,n=0}^{\infty} \frac{(a, m+n)(b, m+n)(c, m)}{(d, 2m+n)m!n!} X^m Y^n \tag{6.1}$$

(see Srivastava and Karlsson (1985) (page 108)).

Horn's method of investigating this convergence, as outlined by Erdélyi (1953) (page 227), is now used. Write (6.1) in the form

$$\sum_{m,n=0}^{\infty} A_{m,n} X^m Y^n \tag{6.2}$$

with

$$f(m, n) = A_{m+1,n}/A_{m,n} = \frac{(a+m+n)(b+m+n)(c+m)}{(d+2m+n)(d+1+2m+n)(1+m)} \tag{6.3}$$

and

$$g(m, n) = A_{m,n+1}/A_{m,n} = \frac{(a+m+n)(b+m+n)}{(d+2m+n)(1+n)}. \tag{6.4}$$

If the power series (6.2) is convergent for $|X|<r$ and $|Y|<s$ and divergent for $|X|>r$ and $|Y|>s$, r and s are referred to as the associated radii of convergence. In the plane (r, s), the points representing r and s lie on a curve C which lies entirely within the rectangle $0<r<R, 0<s<S$. The representation of the domain of convergence of the double series is that region which lies between the curve C and the origin $r=s=0$ inside this rectangle.

If $\Phi(\mu, \nu)=\lim_{t\to\infty} f(\mu t, \nu t)$ and $\Psi(\mu, \nu)=\lim_{t\to\infty} g(\mu t, \nu t)$, then $R=|\Phi(1, 0)|^{-1}$ and $S=|\Psi(0, 1)|^{-1}$ and C has the parametric representation $r=|\Phi(\mu, \nu)|^{-1}$, $s=|\Psi(\mu, \nu)|^{-1}$, where μ and $\nu>0$.

In considering the series G (equation (4.3)), using (6.3) and (6.4), we have

$$\Phi(\mu, \nu)=(\mu+\nu)^2/(2\mu+\nu)^2 \tag{6.5}$$

and

$$\Psi(\mu, \nu)=(\mu+\nu)^2/(2\mu+\nu)/\nu \tag{6.6}$$

so that the Cartesian equation of C is

$$r+s=2r^{1/2} \tag{6.7}$$

and $R=4$ and $S=1$ when

$$|x|<r \qquad \text{and} \qquad |y-1|<s. \tag{6.8}$$

Similarly, for the series H (equation (4.5)) the Cartesian equation of C is

$$r-1/s=2r^{1/2} \tag{6.9}$$

where, in this case,

$$|x/(1-y)^2|<r \qquad \text{and} \qquad |y-1|<s. \tag{6.10}$$

In the case of the series K (equation (4.9)) the cancellation of parameters as used above in connection with the series G reduces this series effectively to F_4, so that the four series comprising (4.10) converge for

$$|\tfrac{1}{2}(1-y)/x^{1/2}|+|\tfrac{1}{2}x^{1/2}|<1. \tag{6.11}$$

The right-hand member of (3.4) converges for

$$|x/y|^{1/2}+|1/y|^{1/2}<1. \tag{6.12}$$

On applying (3.4), in which x and y are interchanged, to the above results, the region of convergence can be extended so that the whole plane is covered. Any region closely neighbouring the singular lines consisting of

$$\begin{array}{lll} x^{1/2}+y^{1/2}=1 & 0<x<1 & 0<y<1 \\ x^{1/2}+1=y^{1/2} & x>1 & y>0 \\ y^{1/2}+1=x^{1/2} & y>1 & x>0 \end{array}$$

involve representations in which convergence is slow.

Appendix: Analytic continuation formulae

The well known result (3.4) is included in the following list for the sake of completeness.

$$\begin{aligned} F_4(a, b; c, c'; x, y) &= \frac{\Gamma(c')\Gamma(b-a)}{\Gamma(c'-a)\Gamma(b)}(-y)^{-a}F_4(a, a-c'+1; c, a-b+1; x/y, 1/y) \\ &\quad +\frac{\Gamma(c')\Gamma(a-b)}{\Gamma(c'-b)\Gamma(a)}(-y)^{-b}F_4(b, b-c'+1; c, b-a+1; x/y, 1/y). \end{aligned} \tag{A.1}$$

$$\begin{aligned} F_4(a, b; c, c'; x, y) &= \frac{\Gamma(c')\Gamma(c'-a-b)}{\Gamma(c'-a)\Gamma(c'-b)}G(a, b, 1+a-c', 1+b-c'; a+b-c'+1, c; x, 1-y) \\ &\quad +\frac{\Gamma(c')\Gamma(a+b-c')}{\Gamma(a)\Gamma(b)}(1-y)^{c'-a-b} \\ &\quad \times H\left(a+b-c', 1+a-c', 1+b-c'; 1+a-c', 1+b-c', c; \frac{x}{(1-y)^2}, y-1\right). \end{aligned} \tag{A.2}$$

$$\begin{aligned} F_4(a, b; c, c'; x, y) &= \frac{\Gamma(c')\Gamma(c'-a-b)}{\Gamma(c'-a)\Gamma(c'-b)}G(a, b, 1+a-c', 1+b-c'; a+b-c'+1, c; x, 1-y) \\ &\quad +L\left(a, b; c, c'; \frac{(1-y)^2}{4x}, \frac{x}{4}\right) \end{aligned} \tag{A.3}$$

where the series L is given by (4.10).

On combining (A.1) with its arguments interchanged on the left-hand side with (A.2) and (A.3), we obtain more results, namely

$$F_4(a, b; c, c'; x, y)$$

$$= \frac{\Gamma(c')\Gamma(b-a)}{\Gamma(c'-a)\Gamma(b)} (-y)^{-a} \left[\frac{\Gamma(c)\Gamma(c+c'-2a-1)}{\Gamma(c-a)\Gamma(c+c'-a-1)} \right.$$

$$\times G\left(a, a-c'+1, a-c+1, 2+a-c-c'; 2a-c-c'+2, a-b+1; \frac{1}{y}, \frac{y-x}{y}\right)$$

$$+ \frac{\Gamma(c)\Gamma(2a-c-c'+1)}{\Gamma(a)\Gamma(a-c'+1)} \left(\frac{y-x}{y}\right)^{c+c'-2a-1}$$

$$\left. \times H\left(\begin{matrix} 2a-c-c'+1, 1+a-c, 2+a-c-c' \\ 1+a-c, 2+a-c-c', a-b+1 \end{matrix} ; \frac{y}{(y-x)^2}, \frac{x-y}{y}\right) \right]$$

$$+ \frac{\Gamma(c')\Gamma(a-b)}{\Gamma(c'-b)\Gamma(a)} (-y)^{-b} \left[\frac{\Gamma(c)\Gamma(c+c'-2b-1)}{\Gamma(c-b)\Gamma(c+c'-b-1)} \right.$$

$$\times G\left(b, b-c'+1, b-c+1, 2+b-c-c'; 2b-c-c'+2, b-a+1; \frac{1}{y}, \frac{y-x}{y}\right)$$

$$+ \frac{\Gamma(c)\Gamma(2b-c-c'+1)}{\Gamma(b)\Gamma(b-c'+1)} \left(\frac{y-x}{y}\right)^{c+c'-2b-1}$$

$$\left. \times H\left(\begin{matrix} 2b-c-c'+1, 1+b-c, 2+b-c-c' \\ 1+b-c, 2+b-c-c', b-a+1 \end{matrix} ; \frac{y}{(y-x)^2}, \frac{x-y}{y}\right) \right] \quad \text{(A.4)}$$

and

$$F_4(a, b; c, c'; x, y)$$

$$= \frac{\Gamma(c')\Gamma(b-a)}{\Gamma(c'-a)\Gamma(b)} (-y)^{-a} \left[\frac{\Gamma(c)\Gamma(c+c'-2a-1)}{\Gamma(c-a)\Gamma(c+c'-a-1)} \right.$$

$$\times G\left(a, a-c'+1, a-c+1, 2+a-c-c'; 2a-c-c'+2, a-b+1; \frac{1}{y}, \frac{y-x}{y}\right)$$

$$\left. + L\left(a, a-c'+1; a-b+1, c; \frac{(y-x)^2}{4y}, \frac{1}{4y}\right) \right]$$

$$+ \frac{\Gamma(c')\Gamma(a-b)}{\Gamma(c'-b)\Gamma(a)} (-y)^{-b} \left[\frac{\Gamma(c)\Gamma(c+c'-2b-1)}{\Gamma(c-b)\Gamma(c+c'-b-1)} \right.$$

$$\times G\left(b, b-c'+1, b-c+1, 2+b-c-c'; 2b-c-c'+2, b-a+1; \frac{1}{y}, \frac{y-x}{y}\right)$$

$$+ L\left(b, b-c'+1; b-a+1; c; \frac{(y-x)^2}{4y}, \frac{1}{4y}\right)\Bigg]. \tag{A.5}$$

The above analytic continuation formulae can easily be implemented by means of a small computer for most values of the parameters. Convergence is sometimes slow near the boundaries of convergence of the series.

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